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# Notes on "Notes on the Gini Coefficient"[1]

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A point of historical interest – undoubtedly arcane to some – is that the esteemed Corrado Gini (Gini 1914) produced no fewer than 13 Gini coefficients. I am not entirely sure what all thirteen are, but two of them seem to have survived the test of time and are widely used. The first is expressed in the formula:

$$G = \frac{1}{2\mu n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \,, \tag{1}$$

where $n$ represents the population, $y$ denotes incomes, and $\mu$, their mean. The second is given by the formula:

$$G' = \frac{1}{2\mu n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| \,, \tag{2}$$

which seems to contain only a minor difference, dividing as it does by $n(n-1)$ instead of $n^2$. As such, $G$ and $G'$ agree ordinally on all comparisons within any given population, but they could disagree across comparisons with populations of varying sizes.

Is this worth a sleepless night or two? Not really, especially if you are not devoted to teaching the subject with the care and precision that my erstwhile student (and now friend) Julia Schwenkenberg brings to it. Julia was teaching her Rutgers students the fundamentals of inequality measurement, when she came across (and pointed out to me) the following example in the Curriculum Open-access Resources in Economics (CORE) textbook: "There are just two individuals in the population and one has all the income . . . [This is] perfect inequality, as we would expect".[2] She went on to observe that the Gini (or perhaps I should say, the Gini as represented in the example) was taken to be equal to its maximum value of 1: a foregone conclusion, seemingly.[3]

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But ought it be? Julia was well aware of the "population neutrality" principle underlying inequality measurement, which stated as an axiom that population cloning of all individuals – while keeping their incomes unchanged – 'should not' change any Lorenz-consistent measure of economic inequality. So the configuration in the text should exhibit no change in inequality if, instead, two people had one unit each of income, and the other two had none. Yet, the latter configuration is surely less unequal than one in which one person had two units of income, and the other three had none. Ergo, the very first situation should be less unequal than the one in which one person out of four (as opposed to one of out of two) had all the income.

Yet, even if we accept this reasoning, should there not be room for the Gini coefficient to rise further from its two-person value of 1? The further question also arises: why is it already at 1?

This issue was intriguing (and confusing) enough for all concerned that Julia wrote me about it, especially since my textbook (Ray 1998) and landmark monographs, such as Sen (1973), use the formula $G$ for the Gini coefficient, whereas the CORE text seemed to be using another formula altogether. After discussions with Rajiv Sethi, and later Sam Bowles and Wendy Carlin, it soon emerged that "the" Gini in the CORE text was actually $G'$. It was a version favoured by two of the CORE authors (Bowles and Carlin 2020), and indeed, it hits its maximum value in the example above. The other one, $G$, does not: it equals $1/2$, well below its maximum value of 1.

Rajiv Sethi's excellent notes on the subject ("Notes on the Gini Coefficient", above), to which this is a response, lay bare the difference. As already noted, both measures (and eleven others, to boot) had been proposed by the prolific Gini, so no claim to true inheritance could be advanced on that somewhat legalistic but otherwise useless basis, "what did Gini really say?" – and fortunately so, for many truths have been trampled underfoot by such convenient excuses. Rather, we must truly evaluate the measures from first principles to advance the discussion, which is what Rajiv accomplishes in large part.

The measure $G$ satisfies all the axioms underlying the Lorenz partial order: population and income invariance, as well as the transfer principle. The Lorenz order, in turn, is at the very heart of inequality measurement and forms a welfare basis for it (see, among others, Atkinson 1983 and Dasgupta *et al.* 1973).[4] $G$ completes this partial order – see Thon (1982) for an axiomatisation. To be sure, it is not the only order that completes the Lorenz: other examples include the coefficient of variation or the Theil index – see Ray (1998) for more on these matters. Nevertheless, $G$ is one of them; it is Lorenz-consistent in the sense of satisfying the axioms that I've just mentioned.

In contrast, as Rajiv explains, $G'$ would highlight the fact that in the two-person example given above, inequality has been stretched to its limit: how can things be any more unequal than already expressed as unequal *in that two-person society*? This brings us to what Rajiv, following Thon (1982), calls strong comparability: "the range spanned by the index depends neither on total income nor on population." In particular, within any society with a fixed population, the index should be able to move from its minimum value (0) to its maximum (1), so that the index recognises clearly the upper limits to inequality *for that fixed population size.* Clearly, $G$ does not do this – it varies only from 0 to 1/2 in a two-person society. Indeed, by the argument given five paragraphs ago, we can already see that *no measure satisfying the population principle and the transfer principle can be strongly comparable.* In a gesture of inclusiveness, Rajiv concludes above: "It is probably best, however, if students are at least made aware of the existence and historical origins of both versions and presented with the arguments in favour of each. There is no uniquely correct Gini coefficient".

Given a choice between two incompatible desiderata, I can sympathise with Rajiv's assertion that there is no "uniquely correct" Gini coefficient. Indeed, given the vast number of Lorenz completions at our disposal, *there is no uniquely correct measure of inequality, even under the Lorenz axioms*, let alone a uniquely correct Gini. Each measure must be evaluated by the core ethical axioms they satisfy, and we then need to refer to our own ethical system to determine which set of axioms fits the best. In this sense, I agree with Rajiv.

That being said, there are axioms, and then there are axioms. I have already mentioned the long and venerable history of the Lorenz curve, dating back to Lorenz (1905). Its foundation is based on a fundamental set of axioms: population neutrality and income neutrality as well as the transfers principle of Pigou and Dalton. These axioms can and have been questioned; for instance, my work with Joan Esteban on the measurement of polarisation (Esteban and Ray 1994) comes from dropping the transfers principle. Nevertheless, as contributions to a *welfare economics* foundation for inequality measurement, these are the key axioms, and all further explorations stem from them – or should.

The fact that $G$ satisfies all three axioms, while $G'$, as noted earlier, fails population neutrality, is an *a priori* (though not yet definitive) cause for suspecting the credentials of $G'$. A noteworthy example comes from Foster (1983, p. 108), who writes down all the axioms (or "properties") that underpin Lorenz, *except for population neutrality*, and then observes of the rest:

> In fact, since each property is a restriction ... in isolation without reference to cross-population comparisons, [they] admit even more measures than Fields and Fei [1978] indicate. Consider the measure that takes the Gini coefficient index at even sized populations, and the coefficient of variation index at odd. *This absurd measure* quite clearly satisfies all [the] properties.
>
> The above example serves to point out the desirability of a property which would coordinate the indices *into one cohesive measure*. Another property [the population principle] suggested by Dalton [1920] does this in a particularly natural way. (*emphases added*).

As Foster correctly notes, the population principle is imposed to make the reader aware that we cannot have potentially unrelated and therefore "absurd" measures on different population layers: they must be connected in a "cohesive" way. Foster goes on to introduce the very population neutrality principle – suggested by Dalton as embedded in the Lorenz curve – that $G$ satisfies and that $G'$ does not. The reason for this difference is that $G'$ refuses to entertain the requirement that its comparisons within a population must be contextualised in the broader space of all populations.

To see directly the failure in the cohesion of $G'$, consider again the CORE textbook example, in which one person gets all the income in a two-person society. According to $G'$ (and the CORE text), this situation is just as unequal as the one in which one person gets all in the income in a million-person society. To me, this is indeed an example of the absurdity that Foster refers to when different population layers are not connected in any coherent way. Taking the extended example a step further, we must conclude that under $G'$, a situation in which two persons share all the income in a million-person society is strictly more equal than the two-person example mentioned in the text. This even more absurd consequence comes from the additional application of the transfer principle, which $G'$ does indeed satisfy.

Of course, $G$ exhibits none of these strange behaviours. It would rank the one-person-in-two-persons example as more equal than the one person-in-a-million example, and the same applies to the two-persons-in-a-million example.

Rajiv would respond that $G$ fails strong comparability: it does not reach 1 when the two-person society is stretched to its unequal limit. Well, I don't see why strong comparability makes sense. Why must a measure declare perfect inequality just because feasibility constrains a particular situation from exhibiting still greater inequality? The latter is a property of the feasible set and should not influence evaluation, just as a utility function is not affected by the budget set on which it operates. To explain why $G'$ also fails this desideratum, consider a two-person society, in which either person could asexually clone themself into two, with further income transfers possible among or across the clones. Such cloning is, however, prohibited, and so all one can actually do in this society is transfer income between two individuals without cloning either of them. Strong comparability then requires an inequality measure to reach its maximum possible value when one person has all the income. Suppose, however, that a divine decree permits cloning; the range of inequality values within the earlier two-person range would need to contract artificially, so as to accommodate the newly unequal possibilities that arise. In short, this represents a case of our measurement indicator responding to the feasible set.

In all of this, I am aware that my arguments do not constitute a logical attack on $G'$ in favour of $G$, in the sense of claiming that there is some failure of Aristotelian logic in the very fabric of $G'$. There is no such attack. Rather, my intention is to appeal to the reader's intuitive sensibilities via the discussion of axioms. In fact, without an axiomatic system, there are no bounds: it's a veritable free-for-all. We would be taking all too literally Sen's beautiful Dedication to his daughters, at the start of *On Economic Inequality* (1973, red. 1997: v):

> *with the hope that when they grow up*
> *they will find less of it no matter*
> *how they decide to measure it*

In a free-for-all, that utopian dream cannot happen, but once constrained by the spirit of a reasonable axiomatic system, it could. That is the spirit in which I reject $G'$ in favour of $G$.

## Notes

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[2] As found in The CORE Team textbook, *The Economy*, Unit 5. Property and power: Mutual gains and conflict, Section 12. Measuring income inequality. For more detail on the CoreEcon project, see https://www.core-econ.org/.

[3] See https://bit.ly/3kB8KNN.

[4] The characterisation by Rothschild and Stiglitz (1970) of "increasing risk" is also relevant here, despite its focus on risk and uncertainty; the two share parallel features.